Special Second Order Equations (Sect. 2.2).

- Special Second order nonlinear equations.
  - Function y missing. (Simpler)
  - Variable t missing. (Harder)
- Reduction order method.

Special Second order nonlinear equations

Definition

Given a functions $f : \mathbb{R}^3 \to \mathbb{R}$, *a second order* differential equation in the unknown function $y : \mathbb{R} \to \mathbb{R}$ is given by

$$y'' = f(t, y, y').$$

The equation is *linear* iff f is linear in the arguments y and y'.

Remarks:

- Nonlinear second order differential equation are usually difficult to solve.
- However, there are two particular cases where *second order* equations can be transformed into *first order* equations.
  (a) $y'' = f(t, y')$. The function y is missing.
  (b) $y'' = f(y, y')$. The independent variable t is missing.

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Special Second order: y missing.

Theorem

If second order differential equation has the form $y'' = f(t, y')$, then the equation for $v = y'$ is the first order equation $v' = f(t, v)$.

Example

Find y solution of the second order nonlinear equation $y'' = -2t\,(y')^2$ with initial conditions $y(0) = 2$, $y'(0) = -1$.

Solution: Introduce $v = y'$. Then $v' = y''$, and

$$v' = -2t\,v^2 \quad \Rightarrow \quad \frac{v'}{v^2} = -2t \quad \Rightarrow \quad -\frac{1}{v} = -t^2 + c.$$

So, $\dfrac{1}{y'} = t^2 - c$, that is, $y' = \dfrac{1}{t^2 - c}$. The initial condition implies

$$-1 = y'(0) = -\frac{1}{c} \quad \Rightarrow \quad c = 1 \quad \Rightarrow \quad y' = \frac{1}{t^2 - 1}.$$

Special Second order: y missing.

Example

Find the y solution of the second order nonlinear equation $y'' = -2t\,(y')^2$ with initial conditions $y(0) = 2$, $y'(0) = -1$.

Solution: Then, $y = \int \frac{dt}{t^2 - 1} + c$. Partial Fractions!

$$\frac{1}{t^2 - 1} = \frac{1}{(t-1)(t+1)} = \frac{a}{(t-1)} + \frac{b}{(t+1)}.$$

Hence, $1 = a(t+1) + b(t-1)$. Evaluating at $t = 1$ and $t = -1$ we get $a = \frac{1}{2}$, $b = -\frac{1}{2}$. So $\frac{1}{t^2-1} = \frac{1}{2}\Big[\frac{1}{(t-1)} - \frac{1}{(t+1)}\Big]$.

$$y = \frac{1}{2}\big(\ln|t-1| - \ln|t+1|\big) + c. \qquad 2 = y(0) = \frac{1}{2}(0 - 0) + c.$$

We conclude $y = \frac{1}{2}\big(\ln|t-1| - \ln|t+1|\big) + 2$. ⊲

Special Second Order Equations (Sect. 2.2).

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  - **Variable t missing. (Harder)**
- Reduction order method.

Special Second order: t missing.

Theorem

Consider a second order differential equation $y'' = f(y, y')$, and introduce the function $v(t) = y'(t)$. If the function y is invertible, then the new function $w(y) = v(t(y))$ satisfies the first order differential equation

$$\frac{dw}{dy} = \frac{1}{w} f(y, w(y)).$$

Proof: Denote $\dot{w} = \frac{dw}{dy}$ and $v' = \frac{dv}{dt}$. Notice that $v'(t) = f(y, v(t))$. By chain rule

$$\dot{w} = \frac{dw}{dy}\Big|_{y} = \frac{dv}{dt}\Big|_{t(y)} \frac{dt}{dy}\Big|_{t(y)} = \frac{v'}{y'}\Big|_{t(y)} = \frac{v'}{v}\Big|_{t(y)} = \frac{f\big(y, v\big)}{v}\Big|_{t(y)}.$$

Therefore, $\dot{w} = f(y, w)/w$. □

Special Second order: t missing.

Example

Find a solution y to the second order equation $y'' = 2y\, y'$.

Solution: The variable t does not appear in the equation.
Hence, $v(t) = y'(t)$. The equation is $v'(t) = 2y(t)\, v(t)$.
Now introduce $w(y) = v(t(y))$. Then

$$\dot{w} = \frac{dw}{dy} = \Big(\frac{dv}{dt}\,\frac{dt}{dy}\Big)\Big|_{t(y)} = \frac{v'}{y'}\Big|_{t(y)} = \frac{v'}{v}\Big|_{t(y)}.$$

Using the differential equation,

$$\dot{w} = \frac{2yv}{v}\Big|_{t(y)} \quad \Rightarrow \quad \dot{w} = 2y \quad \Rightarrow \quad \hat{v}(y) = y^2 + c.$$

Since $v(t) = w(y(t))$, we get $v(t) = y^2(t) + c$.

Special Second order: t missing.

Example

Find a solution y to the second order equation $y'' = 2y\,y'$.

Solution: Recall: $v(t) = y^2(t) + c$. This is a separable equation,

$$\frac{y'(t)}{y^2(t) + c} = 1.$$

Since we only need to find a solution of the equation, and the integral depends on whether $c > 0$, $c = 0$, $c < 0$, we choose (for no special reason) only one case, $c = 1$.

$$\int \frac{dy}{1 + y^2} = \int dt + c_0 \quad \Rightarrow \quad \arctan(y) = t + c_0 y(t) = \tan(t + c_0).$$

Again, for no reason, we choose $c_0 = 0$, and we conclude that one possible solution to our problem is $y(t) = \tan(t)$. $\triangleleft$

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- **Reduction order method.**

Reduction of the order method

Remark: Knowing one solution to a differential equation is enough to find a second solution not proportional to the first one.

Theorem

If a non-zero function y_1 is solution to

$$y'' + p(t)\, y' + q(t)\, y = 0. \tag{1}$$

where p, q are given functions, then a second solution to this same equation is given by

$$y_2(t) = y_1(t) \int \frac{e^{-P(t)}}{y_1^2(t)}\, dt, \tag{2}$$

where $P(t) = \int p(t)\, dt$. Furthermore, y_1 and y_2 are fundamental solutions to Eq. (1).

Reduction of the order method

Example

Find a fundamental set of solutions to

$$t^2 y'' + 2ty' - 2y = 0,$$

knowing that $y_1(t) = t$ is a solution.

Solution: Express $y_2(t) = v(t)\, y_1(t)$. The equation for v comes from $t^2 y_2'' + 2t y_2' - 2y_2 = 0$. We need to compute

$$y_2 = v\, t, \qquad y_2' = t\, v' + v, \qquad y_2'' = t\, v'' + 2v'.$$

So, the equation for v is given by

$$t^2\big(t\, v'' + 2v'\big) + 2t\big(t\, v' + v\big) - 2t\, v = 0$$

$$t^3\, v'' + (2t^2 + 2t^2)\, v' + (2t - 2t)\, v = 0$$

$$t^3\, v'' + (4t^2)\, v' = 0 \quad \Rightarrow \quad v'' + \frac{4}{t} v' = 0.$$

Reduction of the order method

Example

Find a fundamental set of solutions to

$$t^2 y'' + 2ty' - 2y = 0,$$

knowing that $y_1(t) = t$ is a solution.

Solution: Recall: $v'' + \frac{4}{t} v' = 0.$

This is a first order equation for $w = v'$, given by $w' + \frac{4}{t} w = 0$, so

$$\frac{w'}{w} = -\frac{4}{t} \;\Rightarrow\; \ln(w) = -4\ln(t) + c_0 \;\Rightarrow\; w(t) = c_1 t^{-4}, \; c_1 \in \mathbb{R}.$$

Integrating w we obtain v, that is, $v = c_2 t^{-3} + c_3$, with $c_2, c_3 \in \mathbb{R}$.
Recalling that $y_2 = t\, v$ we then conclude that $y_2 = c_2 t^{-2} + c_3 t$.
Choosing $c_2 = 1$ and $c_3 = 0$ we obtain the fundamental solutions
$y_1(t) = t$ and $y_2(t) = \frac{1}{t^2}$. $\triangleleft$

Reduction of the order method

Proof of the Theorem: The choice of $y_2 = v y_1$ implies

$$y_2' = v'\, y_1 + v\, y_1', \qquad y_2'' = v''\, y_1 + 2v'\, y_1' + v\, y_1''.$$

This information introduced into the differential equation says that

$$(v''\, y_1 + 2v'\, y_1' + v\, y_1'') + p\,(v'\, y_1 + v\, y_1') + qv\, y_1 = 0$$

$$y_1\, v'' + (2y_1' + p\, y_1)\, v' + (y_1'' + p\, y_1' + q\, y_1)\, v = 0.$$

The function y_1 is solution of $y_1'' + p\, y_1' + q\, y_1 = 0.$

Then, the equation for v is given by

$$y_1\, v'' + (2y_1' + p\, y_1)\, v' = 0.$$

Reduction of the order method

Recall: $y_1\, v'' + (2y_1' + p\, y_1)\, v' = 0.$

This is a first order eq. for $w(t) = v'(t)$. That is,

$$w' + \Big(2\frac{y_1'}{y_1} + p\Big)\, w = 0.$$

This is the origin of hte name: *Reduction of order method*.
Integrating factor: $\mu = y_1^2\, e^P$, with $P' = p$. Then

$$(y_1^2\, e^P\, w)' = 0 \quad \Rightarrow \quad w = w_0 e^{-P}/y_1^2 \quad \text{choose } w_0 = 1.$$

Then $v' = e^{-P}/y_1^2$, hence

$$v(t) = \int \frac{e^{-P}}{y_1^2}\, dt \quad \Rightarrow \quad y_2(t) = y_1(t) \int \frac{e^{-P(t)}}{y_1^2(t)}\, dt$$

Reduction of the order method

Proof: Recall $y_1\, v'' + (2y_1' + p\, y_1)\, v' = 0$. We now need to show that y_1 and $y_2 = vy_1$ are linearly independent.

$$W_{y_1 y_2} = \begin{vmatrix} y_1 & vy_1 \\ y_1' & (v'y_1 + vy_1') \end{vmatrix} = y_1(v'y_1 + vy_1') - vy_1 y_1'.$$

We obtain $W_{y_1 y_2} = v' y_1^2$. Recall we have $w = v'$,

$$v' = w = e^{-P}/y_1^2 \quad \Rightarrow \quad y_1^2\, v' = e^{-P}$$

Recall that P is a primitive of p, that is, $P'(t) = p(t)$, then we obtain

$$W_{y_1 y_2} = e^{-P},$$

which is non-zero. We conclude that y_1 and $y_2 = vy_1$ are linearly independent. $\square$